HEISENBERG DOUBLE $\mathcal{H}(B^*)$ AS A BRAIDED COMMUTATIVE YETTER–DRINFELD MODULE ALGEBRA OVER THE DRINFELD DOUBLE

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ABSTRACT. We study the Yetter–Drinfeld $\mathcal{D}(B)$-module algebra structure on the Heisenberg double $\mathcal{H}(B^*)$ endowed with a "heterotic" action of the Drinfeld double $\mathcal{D}(B)$. This action can be interpreted in the spirit of Lu's description of $\mathcal{H}(B^*)$ as a twist of $\mathcal{D}(B)$. In terms of the braiding of Yetter–Drinfeld modules, $\mathcal{H}(B^*)$ is braided commutative. By the Brzeziński–Militaru theorem, $\mathcal{H}(B^*) \# \mathcal{D}(B)$ is then a Hopf algebroid over $\mathcal{H}(B^*)$. For B a particular Taft Hopf algebra at a $2p$th root of unity, the construction is adapted to yield Yetter–Drinfeld module algebras over the $2p^3$-dimensional quantum group $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$. In particular, it follows that $\mathrm{Mat}_p(\mathbb{C})$ is a braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-module algebra and $\mathrm{Mat}_p(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2))$ is a Hopf algebroid over $\mathrm{Mat}_p(\mathbb{C})$.

1. INTRODUCTION

For a Hopf algebra B, the Heisenberg double $\mathcal{H}(B^*)$ is the smash product $B^* \# B$ with respect to the left regular action $b \rightharpoonup \beta = \langle \beta'', b\rangle \beta'$ of B on B^*; the composition in $\mathcal{H}(B^*)$ is given by

$$(\alpha \# a)(\beta \# b) = \alpha(a' \rightharpoonup \beta) \# a''b, \qquad \alpha, \beta \in B^*, \quad a, b \in B. \tag{1.1}$$

Let $\mathcal{D}(B)$ be the Drinfeld double of B, with its elements written as $\mu \otimes m$, where $\mu \in B^*$ and $m \in B$; the composition in $\mathcal{D}(B)$ is $(\mu \otimes m)(\nu \otimes n) = \mu(m' \rightharpoonup \nu \leftharpoonup S^{-1}(m''')) \otimes m''n$ (and the coalgebra structure is that of $B^{*\mathrm{cop}} \otimes B$). We define a $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$ as

$$\begin{aligned}(\mu \otimes m) \triangleright (\beta \# b) = \mu'''(m' \rightharpoonup \beta) S^{*-1}(\mu'') \# \big((m'' b S(m''')) \leftharpoonup S^{*-1}(\mu')\big),\\ \mu \otimes m \in \mathcal{D}(B), \quad \beta \# b \in \mathcal{H}(B^*),\end{aligned} \tag{1.2}$$

where $b \leftharpoonup \mu = \langle \mu, b'\rangle b''$ is the right regular action of B^* on B (and $\langle\, , \rangle$ is the evaluation).

1.1. Theorem. *For a Hopf algebra B with bijective antipode, $\mathcal{H}(B^*)$ endowed with action* (1.2) *and the coaction*

$$\delta : \begin{aligned}&\mathcal{H}(B^*) \to \mathcal{D}(B) \otimes \mathcal{H}(B^*)\\ &\beta \# b \mapsto (\beta'' \otimes b') \otimes (\beta' \# b'')\end{aligned} \tag{1.3}$$

is a (*left–left*) *Yetter–Drinfeld $\mathcal{D}(B)$-module algebra.*

By a Yetter–Drinfeld module algebra we mean a module comodule algebra that is also a Yetter–Drinfeld module, i.e., a compatibility condition between the action and the coaction holds in the form

(1.4) $$(M' \triangleright A)_{(-1)}M'' \otimes (M' \triangleright A)_{(0)} = M'A_{(-1)} \otimes (M'' \triangleright A_{(0)}),$$

where, in our case, $M \in \mathcal{D}(B)$ and $A \in \mathcal{H}(B^*)$.[1]

We recall from [1, 2, 3] that for a Hopf algebra H, a left H-module and left H-comodule algebra X is said to be *braided commutative* (or H-commutative) if

(1.5) $$yx = (y_{(-1)} \triangleright x)y_{(0)}, \qquad x, y \in X.$$

Also, for any two (left–left) Yetter–Drinfeld H-module algebras X and Y, their *braided product* $X \bowtie Y$ is defined as the tensor product with the composition

(1.6) $$(x \bowtie y)(v \bowtie u) = x(y_{(-1)} \triangleright v) \bowtie y_{(0)}u, \quad x, v \in X, \quad y, u \in Y.$$

(This gives a Yetter–Drinfeld module algebra.)

1.2. Theorem. *$\mathcal{H}(B^*)$ is a braided ($\mathcal{D}(B)$-) commutative algebra. Moreover, $\mathcal{H}(B^*)$ is the braided product*

$$\mathcal{H}(B^*) = B^{*\mathrm{cop}} \bowtie B,$$

where $B^{*\mathrm{cop}}$ and B are (braided commutative) Yetter–Drinfeld $\mathcal{D}(B)$ module algebras by restriction, i.e., with the $\mathcal{D}(B)$ action

$$(\mu \otimes m) \triangleright \beta = \mu''(m \rightharpoonup \beta)S^{*-1}(\mu'), \qquad (\mu \otimes m) \triangleright b = (m'bS(m'')) \leftharpoonup S^{*-1}(\mu)$$

and coaction $\delta : \beta \mapsto (\beta'' \otimes 1) \otimes \beta'$, $\delta : b \mapsto (\varepsilon \otimes b') \otimes b''$ ($\beta \in B^*$, $b \in B$).

1.2.1. As a corollary, the Brzeziński–Militaru theorem [2] then "provides one with a rich source of examples of bialgebroids." In particular, *for any Hopf algebra B with bijective antipode, the "quadruple" $\mathcal{H}(B^*) \# \mathcal{D}(B)$*, where the smash product is defined with respect to action (1.2), *is a Hopf algebroid over $\mathcal{H}(B^*)$.*

1.2.2. A "pseudoadjoint" interpretation of (1.2). The $\mathcal{D}(B)$-action (1.2) first appeared in [4]. To borrow a popular term from string theory [5] (where it was also a borrowing originally), this action may be termed "heterotic" because it is constructed by combining left and right $\mathcal{D}(B)$ actions, as we describe in **2.2.2** (and the heterotic string famously combines "left" and "right"). Or because (1.2) "cross-breeds" regular and adjoint actions.

Trying to quantify how "far" (1.2) is from the adjoint action, we arrive at a useful interpretation of our "heterotic" action by extending Lu's description of the product on $\mathcal{H}(B^*)$ as a twist of the product on $\mathcal{D}(B)$ [6]. The two algebraic structures, $\mathcal{D}(B)$ and $\mathcal{H}(B^*)$, are defined on the same vector space $B^* \otimes B$, and the product (1.1) in $\mathcal{H}(B^*)$, temporarily denoted by $\star$, can be written as

(1.7) $$M \star N = M'N'\eta(M'', N''), \qquad M, N \in \mathcal{D}(B)$$

for a certain 2-cocycle $\eta : \mathcal{D}(B) \otimes \mathcal{D}(B) \to k$ [6]. In the same vein, the $\mathcal{D}(B)$ action on

[1]For a Hopf algebra H and a left H-comodule X, we write the coaction $\delta : X \to H \otimes X$ as $\delta(x) = x_{(-1)} \otimes x_{(0)}$; then the comodule axioms are $\langle \varepsilon, x_{(-1)} \rangle x_{(0)} = x$ and $x'_{(-1)} \otimes x''_{(-1)} \otimes x_{(0)} = x_{(-1)} \otimes x_{(0)\,(-1)} \otimes x_{(0)\,(0)}$.

$\mathcal{H}(B^*)$ in (1.2) can be rewritten in the "pseudoadjoint" form

$$(M,A) \mapsto M' \star A \star \mathsf{s}(M''), \qquad M \in \mathcal{D}(B), \quad A \in \mathcal{H}(B^*), \tag{1.8}$$

where $\mathsf{s}(M) = \eta(M', M'')S(M''')$. Some "antipode-like" properties of s allow independently verifying that the right-hand side here *is* an action, as we show in **2.4.3**, where further details are given.

1.2.3. The Heisenberg double $\mathcal{H}(B^*) = B^{*\mathrm{cop}} \bowtie B$ can be regarded as the lowest term, $\mathcal{H}(B^*) = \mathfrak{H}_2$, in a series of *Heisenberg n-tuples,* or *chains* $\mathfrak{H}_n$ — the Yetter–Drinfeld $\mathcal{D}(B)$-modules

$$\mathfrak{H}_{2n} = B^{*\mathrm{cop}} \bowtie B \bowtie B^{*\mathrm{cop}} \bowtie B \bowtie \ldots \bowtie B,$$

$$\mathfrak{H}_{2n+1} = B^{*\mathrm{cop}} \bowtie B \bowtie B^{*\mathrm{cop}} \bowtie B \bowtie \ldots \bowtie B \bowtie B^{*\mathrm{cop}}$$

(with $2n$ and $2n+1$ factors), with the relations

$$b[2i]\,\beta[2j+1] = (b' \rightharpoonup \beta)[2j+1]\,b''[2i] \quad \text{for all } i \text{ and } j, \tag{1.9}$$

(where $B^{*\mathrm{cop}} \to B^{*\mathrm{cop}}[2j+1]$ and $B \to B[2i]$ are the morphisms onto the respective factors, and we omit $\bowtie$ for simplicity), and

$$\alpha[2i+1]\,\beta[2j+1] = (\alpha'''\beta S^{*-1}(\alpha''))[2j+1]\,\alpha'[2i+1], \quad i \geqslant j \tag{1.10}$$

$$a[2i]\,b[2j] = (a'bS(a''))[2j]\,a'''[2i], \quad i \geqslant j, \tag{1.11}$$

where $a, b \in B$, $\alpha, \beta \in B^{*\mathrm{cop}}$.

1.3. As regards the popular subject of Yetter–Drinfeld modules, we note Refs. [8, 9, 10, 11, 12, 1]. Heisenberg doubles [13, 14, 15, 6], among various smash products, have attracted some attention, notably in relation to Hopf algebroid constructions [16, 17, 2] (the basic observation being that $\mathcal{H}(B^*)$ is a Hopf algebroid over B^* [16]) and also from various other standpoints and for different purposes [18, 7, 19, 20]. (A relatively recent paper where Yetter–Drinfeld-like structures are studied in relation to "smash" products is [21].)

1.4. The above results are proved quite straightforwardly. The proofs are given in Sec. 2; there, B denotes a Hopf algebra with bijective antipode. When we pass to an example in Sec. 3, B becomes a particular Taft Hopf algebra.

1.5. The example worked out in Sec. 3 is that of the $2p^3$-dimensional quantum group $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ at the $2p$th root of unity

$$\mathfrak{q} = e^{\frac{i\pi}{p}},$$

($p = 2, 3, \ldots$). This is the Hopf algebra with generators E, K, and F and the relations

$$KEK^{-1} = \mathfrak{q}^2 E, \quad KFK^{-1} = \mathfrak{q}^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

$$E^p = F^p = 0, \quad K^{2p} = 1,$$

and the Hopf algebra structure $\Delta(E) = E\otimes K + 1\otimes E$, $\Delta(K) = K\otimes K$, $\Delta(F) = F\otimes 1 + K^{-1}\otimes F$, $\varepsilon(E) = \varepsilon(F) = 0$, $\varepsilon(K) = 1$, $S(E) = -EK^{-1}$, $S(K) = K^{-1}$, $S(F) = -KF$.

1.5.1. $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ is "almost" the Drinfeld double of a $4p^2$-dimensional Taft Hopf algebra B, more precisely, a "truncation" of the double obtained by taking a quotient and then restricting to a subalgebra. This close kinship of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ to a Drinfeld double extends to the "Heisenberg side": it turns out that the pair $(\mathcal{D}(B),\mathcal{H}(B^*))$ can also be "truncated" to a pair $(\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2),\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2))$ of $2p^3$-dimensional algebras, where $\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2)$ is a braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$-module algebra.

1.5.2. Interestingly, the $2p^3$-dimensional braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$-module algebra $\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2)$ can be described as

$$\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2) \cong \mathrm{Mat}_p\bigl(\mathbb{C}_{2p}[\lambda]\bigr), \qquad \mathbb{C}_{2p}[\lambda] \equiv \mathbb{C}[\lambda]/(\lambda^{2p}-1),$$

which adds a matrix flavor to our example. In the matrix language, the relevant structures are described as follows.

First, the $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ action on matrices $X = (x_{ij})$ with λ-dependent entries is given by

$$(K\triangleright X)_{ij} = \mathfrak{q}^{2(i-j)}\bigl(x_{ij}\big|_{\lambda\to\mathfrak{q}^{-1}\lambda}\bigr), \tag{1.12}$$

and

$$E\triangleright X = \frac{1}{\mathfrak{q}-\mathfrak{q}^{-1}}\bigl(XZ - Z(K\triangleright X)\bigr), \tag{1.13}$$

$$F\triangleright X = \frac{1}{\mathfrak{q}-\mathfrak{q}^{-1}}\bigl(DX - (K^{-1}\triangleright X)D\bigr), \tag{1.14}$$

where

$$Z = \begin{pmatrix} 0 & \dots\dots\dots\dots & & & 0\\ 1 & 0 & \dots\dots\dots & & 0\\ 0 & 1 & 0 & \dots & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & \dots\dots & & 1 & 0 \end{pmatrix}, \qquad D = (\mathfrak{q}-\mathfrak{q}^{-1})\begin{pmatrix} 0 & 1 & \dots\dots\dots\dots & & 0\\ 0 & 0 & \mathfrak{q}^{-1}[2] & \dots & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & \dots\dots\dots & & 0 & \mathfrak{q}^{2-p}[p-1]\\ 0 & \dots\dots\dots\dots & & & 0 \end{pmatrix} \tag{1.15}$$

and we use the standard notation

$$[n] = \frac{\mathfrak{q}^n-\mathfrak{q}^{-n}}{\mathfrak{q}-\mathfrak{q}^{-1}}, \quad [n]! = [1][2]\dots[n], \quad \begin{bmatrix} m\\ n \end{bmatrix} = \frac{[m]!}{[m-n]!\,[n]!}.$$

Next, to describe the coaction $\delta : \mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda]) \to \overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)\otimes\mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda])$, we first note that $\mathbb{C}_{2p}[\lambda]$ is the algebra of coinvariants, $\delta : \lambda \mapsto 1\otimes\lambda$. It therefore remains to define δ on "constant" matrices $\mathrm{Mat}_p(\mathbb{C})$. But the full matrix algebra $\mathrm{Mat}_p(\mathbb{C})$ is algebraically generated by the above Z and D, and we have

$$\delta : \begin{aligned} Z &\mapsto K^{-1}\otimes Z - (\mathfrak{q}-\mathfrak{q}^{-1})EK^{-1}\otimes 1,\\ D &\mapsto K^{-1}\otimes D + (\mathfrak{q}-\mathfrak{q}^{-1})F\otimes 1. \end{aligned} \tag{1.16}$$

To summarize,

1.5.3. Theorem. *With the above* $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ *action and coaction,* $\mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda])$ *and* $\mathrm{Mat}_p(\mathbb{C})$ *are braided commutative left–left Yetter–Drinfeld* $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$*-module algebras.*

1.5.4. By Theorem 4.1 in [2], as already noted in **1.2.1**, we then have examples of bialgebroids:

$$\mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda]) \# \overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2) \ \text{ and } \ \mathrm{Mat}_p(\mathbb{C}) \# \overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$$

are Hopf algebroids over the respective algebras $\mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda])$ and $\mathrm{Mat}_p(\mathbb{C})$; further details are given in **3.4**.

1.6. Hopf algebras and logarithmic conformal field theory. An additional source of interest in $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ is its occurrence in a version of the Kazhdan–Lusztig duality [22], specifically, as the quantum group dual to a class of *logarithmic models* of conformal field theory [23, 24, 25, 26, 27].

In the "logarithmic" Kazhdan–Lusztig duality, $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ appeared in [23, 24]; subsequently, it gradually transpired (with the final picture having emerged from [28]) that that was just a continuation of a series of previous (re)discoveries of this quantum group [29, 30, 31] (also see [32]). The ribbon and (somewhat stretching the definition) factorizable structures of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ were worked out in [23].

That $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ is Kazhdan–Lusztig-dual to logarithmic models of conformal field theory — specifically, $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ at $\mathfrak{q} = e^{\frac{i\pi}{p}}$ is dual to the $(p,1)$ logarithmic model [33] — means several things, in particular, (i) the $SL(2,\mathbb{Z})$ representation on the $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ center coincides with the $SL(2,\mathbb{Z})$ representation generated from the characters of the symmetry algebra of the logarithmic model [23], the so-called triplet $W(p)$ algebra [34, 33, 35, 36, 37, 38], and (ii) the $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ and $W(p)$ representation categories are equivalent [24, 26, 27].

The "Heisenberg counterpart" of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, its braided commutative Yetter–Drinfeld module algebra $\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2)$, is also likely to play a role in the Kazhdan–Lusztig context [39, 4], but this is a subject of future work.

2. $\mathcal{H}(B^*)$ AS A YETTER–DRINFELD $\mathcal{D}(B)$-MODULE ALGEBRA

We begin with simple facts about Yetter–Drinfeld module algebras, concentrating in **2.1** on the construction of a braided commutative Yetter–Drinfeld module algebra as a braided product $X \bowtie Y$ of two such algebras X and Y. In **2.2**, we then specialize to $X = B^{*\mathrm{cop}}$ and $Y = B$, viewed as $\mathcal{D}(B)$ module algebras under the heterotic action. We verify that all the necessary conditions are then satisfied, hence our conclusion in **2.3**. In **2.4**, we give a "pseudoadjoint" interpretation of the heterotic action, and in **2.5** consider multiple "alternating" braided products.

2.1. The category of Yetter–Drinfeld modules over a Hopf algebra with bijective antipode is well known to be braided, with the braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ given by

$$c_{X,Y} : x \otimes y \mapsto (x_{(-1)} \triangleright y) \otimes x_{(0)}.$$

The inverse is $c^{-1}_{X,Y} : y \otimes x \mapsto x_{(0)} \otimes S^{-1}(x_{(-1)}) \triangleright y$.

We say that two Yetter–Drinfeld modules X and Y are *braided symmetric* if

$$c_{Y,X} = c^{-1}_{X,Y}$$

(note that both sides here are maps $Y \otimes X \to X \otimes Y$), that is,

$$(y_{(-1)} \triangleright x) \otimes y_{(0)} = x_{(0)} \otimes \big(S^{-1}(x_{(-1)}) \triangleright y\big).$$

2.1.1. Lemma. *Let X and Y be braided symmetric Yetter–Drinfeld modules, each of which is a braided commutative Yetter–Drinfeld module algebra. Then their braided product $X \bowtie Y$ is a braided commutative Yetter–Drinfeld module algebra.*

2.1.2. Proof. Beyond the standard facts, we have to show the braided commutativity, i.e.,

$$\big((x \bowtie y)_{(-1)} \triangleright (v \bowtie u)\big)(x \bowtie y)_{(0)} = (x \bowtie y)(v \bowtie u) \tag{2.1}$$

for all $x, v \in X$ and $y, u \in Y$. For this, we write the condition $c_{X,Y} = c^{-1}_{Y,X}$ as

$$(x_{(-1)} \triangleright y) \otimes x_{(0)} = y_{(0)} \otimes \big(S^{-1}(y_{(-1)}) \triangleright x\big)$$

and use this to establish an auxiliary identity,

$$\begin{aligned}\big((x_{(-1)} \triangleright y)_{(-1)} \triangleright x_{(0)}\big) \otimes (x_{(-1)} \triangleright y)_{(0)} &= \big(y_{(0)\,(-1)} \triangleright \big(S^{-1}(y_{(-1)}) \triangleright x\big)\big) \otimes y_{(0)\,(0)} \\ &= \big(y''_{(-1)} S^{-1}(y'_{(-1)}) \triangleright x\big) \otimes y_{(0)} \\ &= x \otimes y.\end{aligned} \tag{2.2}$$

The left-hand side of (2.1) can then be calculated as

$$\begin{aligned}&\big((x \bowtie y)_{(-1)} \triangleright (v \bowtie u)\big)(x \bowtie y)_{(0)} \\ &\quad = \big(x_{(-1)} y_{(-1)} \triangleright (v \bowtie u)\big)(x_{(0)} \bowtie y_{(0)}) \\ &\quad = \big((x'_{(-1)} y'_{(-1)} \triangleright v) \bowtie (x''_{(-1)} y''_{(-1)} \triangleright u)\big)(x_{(0)} \bowtie y_{(0)}) \\ &\quad = (x'_{(-1)} y'_{(-1)} \triangleright v)\big((x''_{(-1)} y''_{(-1)} \triangleright u)_{(-1)} \triangleright x_{(0)}\big) \bowtie (x''_{(-1)} y''_{(-1)} \triangleright u)_{(0)} y_{(0)} \\ &\quad = (x_{(-1)} y'_{(-1)} \triangleright v)\big((x_{(0)\,(-1)} \triangleright (y''_{(-1)} \triangleright u))_{(-1)} \triangleright x_{(0)\,(0)}\big) \bowtie (x_{(0)\,(-1)} \triangleright (y''_{(-1)} \triangleright u))_{(0)} y_{(0)} \\ &\quad = (x_{(-1)} y'_{(-1)} \triangleright v) x_{(0)} \bowtie (y''_{(-1)} \triangleright u) y_{(0)},\end{aligned}$$

just because of (2.2) in the last equality. But the right-hand side of (2.1) is

$$\begin{aligned}(x \bowtie y)(v \bowtie u) &= x(y_{(-1)} \triangleright v) \bowtie y_{(0)} u \\ &= (x_{(-1)} y_{(-1)} \triangleright v) x_{(0)} \bowtie (y_{(0)\,(-1)} \triangleright u) y_{(0)\,(0)}\end{aligned}$$

because X and Y are both braided commutative. The two expressions coincide.

2.1.3. Remark. Because the braided symmetry condition is symmetric with respect to the two modules, we also have the braided symmetric Yetter–Drinfeld module algebra $Y \bowtie X$, with the product

$$(y \bowtie x)(u \bowtie v) = y(x_{(-1)} \triangleright u) \bowtie x_{(0)} v.$$

In addition to the multiplication inside Y and inside X, this formula expresses the relations $xu = (x_{(-1)} \triangleright u)x_{(0)}$ satisfied in $Y \bowtie X$ by $x \in X$ and $u \in Y$. Because $c_{X,Y} = c^{-1}_{Y,X}$, these are the same relations $ux = (u_{(-1)} \triangleright x)u_{(0)}$ that we have in $X \bowtie Y$. Somewhat more formally, the isomorphism

$$\phi : X \bowtie Y \to Y \bowtie X$$

is given by $\phi : x \bowtie y \mapsto (x_{(-1)} \triangleright y) \bowtie x_{(0)}$. This is a module map by virtue of the Yetter–Drinfeld condition, and it is immediate to verify that $\delta(\phi(x \bowtie y)) = (\mathrm{id} \otimes \phi)(\delta(x \bowtie y))$. That ϕ is an algebra map follows by calculating

$$\begin{aligned}
\phi(x \bowtie y)\phi(v \bowtie u) &= \big((x_{(-1)} \triangleright y) \bowtie x_{(0)}\big)\big((v_{(-1)} \triangleright u) \bowtie v_{(0)}\big)\\
&= (x_{(-1)} \triangleright y)(x_{(0)(-1)} v_{(-1)} \triangleright u) \bowtie x_{(0)(0)} v_{(0)}\\
&= (x'_{(-1)} \triangleright y)(x''_{(-1)} v_{(-1)} \triangleright u) \bowtie x_{(0)} v_{(0)}\\
&= x_{(-1)} \triangleright \big(y(v_{(-1)} \triangleright u)\big) \bowtie x_{(0)} v_{(0)}
\end{aligned}$$

and

$$\begin{aligned}
\phi\big((x \bowtie y)(v \bowtie u)\big) &= \phi\big(x(y_{(-1)} \triangleright v) \bowtie y_{(0)} u\big)\\
&= \big(x_{(-1)}(y_{(-1)} \triangleright v)_{(-1)} \triangleright (y_{(0)} u)\big) \bowtie x_{(0)} (y_{(-1)} \triangleright v)_{(0)}\\
&\overset{\checkmark}{=} x_{(-1)} \triangleright (y_{(0)} u)_{(0)} \bowtie x_{(0)} \big(S^{-1}(y_{(0)(-1)} u_{(-1)}) \triangleright (y_{(-1)} \triangleright v)\big)\\
&= x_{(-1)} \triangleright (y_{(0)} u_{(0)}) \bowtie x_{(0)} \big(S^{-1}(y''_{(-1)} u_{(-1)}) y'_{(-1)} \triangleright v\big)\\
&= x_{(-1)} \triangleright (y u_{(0)}) \bowtie x_{(0)} \big(S^{-1}(u_{(-1)}) \triangleright v\big)\\
&\overset{\checkmark}{=} x_{(-1)} \triangleright \big(y(v_{(-1)} \triangleright u)\big) \bowtie x_{(0)} v_{(0)},
\end{aligned}$$

where the braided symmetry condition was used in each of the $\overset{\checkmark}{=}$ equalities.

2.2. We intend to use **2.1.1** in the case where $X = B^{*\mathrm{cop}}$ and $Y = B$. This requires some preparations.

2.2.1. Lemma. *For a Hopf algebra B with bijective antipode, the formulas*

$$(\mu \otimes m) \triangleright \beta = \mu''(m \rightharpoonup \beta) S^{*-1}(\mu'), \qquad (\mu \otimes m) \triangleright b = (m' b S(m'')) \leftharpoonup S^{*-1}(\mu)$$

make $B^{*\mathrm{cop}}$ and B into left $\mathcal{D}(B)$-module algebras.

2.2.2. This is known, e.g., from [17], where both these actions are discussed and references to the previous works are given. The $\mathcal{D}(B)$ action on B^* is obtained by restricting

the *left* regular action of $\mathcal{D}(B)$ on $\mathcal{D}(B)^* \cong B\otimes B^*$ [6],

$$(\mu\otimes m)\rightharpoonup(b\otimes\beta) = (\mu''\rightharpoonup b)\otimes\mu'''(m\rightharpoonup\beta)S^{*-1}(\mu'),$$

to $1\otimes B^*$. Similarly, the $\mathcal{D}(B)$ action on B is obtained [40] by restricting the *right* regular action of $\mathcal{D}(B)$ on $\mathcal{D}(B)^* \cong B\otimes B^*$ to $B\otimes\varepsilon$ and using the antipode to convert it into a left action. The right regular action of $\mathcal{D}(B)$ on $\mathcal{D}(B)^*$ is [6, 17]

$$(b\otimes\beta)\leftharpoonup(\mu\otimes m) = S^{-1}(m''')(b\leftharpoonup\mu)m'\otimes(\beta\leftharpoonup m''),$$

where $\beta\leftharpoonup m = \langle\beta', m\rangle\beta''$ is the right regular action of B on B^*. Restricting to B and replacing $\mu\otimes m$ with $(S(m''')\rightharpoonup S^{*-1}(\mu)\leftharpoonup m')\otimes S(m'')$ then gives the second formula in the lemma.

The following statement is obvious.

2.2.3. Lemma. *With the respective coactions*

$$\delta:\beta\mapsto(\beta''\otimes 1)\otimes\beta', \qquad \delta: b\mapsto(\varepsilon\otimes b')\otimes b'',$$

$B^{*\mathrm{cop}}$ *and* B *are* $\mathcal{D}(B)$*-comodule algebras.*

2.2.4. Lemma. *With the action and coaction in **2.2.1** and **2.2.3**, both* $B^{*\mathrm{cop}}$ *and* B *are Yetter–Drinfeld module algebras.*

It only remains to verify the Yetter–Drinfeld condition in each case. For $B^{*\mathrm{cop}}$, we calculate the left-hand side of (1.4) with $M = \mu\otimes m$ as

$$\begin{aligned}
&((\mu''\otimes m')\triangleright\beta)_{(-1)}(\mu'\otimes m'')\otimes((\mu''\otimes m')\triangleright\beta)_{(0)}\\
&\quad= \big((\mu'''(m'\rightharpoonup\beta)S^{*-1}(\mu''))''\mu'\otimes m''\big)\otimes(\mu'''(m'\rightharpoonup\beta)S^{*-1}(\mu''))'\\
&\quad= \big(\mu^{(5)}(m'\rightharpoonup\beta'')S^{*-1}(\mu^{(2)})\mu^{(1)}\otimes m''\big)\otimes\mu^{(4)}\beta'S^{*-1}(\mu^{(3)})\\
&\quad= \big(\mu'''(m'\rightharpoonup\beta'')\otimes m''\big)\otimes\mu''\beta'S^{*-1}(\mu'),
\end{aligned}$$

but the right-hand side of (1.4) is

$$\begin{aligned}
&(\mu\otimes m)'\beta_{(-1)}\otimes\big((\mu\otimes m)''\triangleright\beta_{(0)}\big)\\
&\quad= (\mu'''\otimes m')(\beta''\otimes 1)\otimes\big(\mu''(m''\rightharpoonup\beta')S^{*-1}(\mu')\big)\\
&\quad= (\mu'''\otimes m')((\beta''\leftharpoonup m'')\otimes 1)\otimes\mu''\beta'S^{*-1}(\mu')\\
&\qquad\text{(because } \beta''\otimes(m\rightharpoonup\beta') = (\beta''\leftharpoonup m)\otimes\beta'\text{)}\\
&\quad=\big(\mu'''(m^{(1)}\rightharpoonup\beta''\leftharpoonup m^{(4)}S^{-1}(m^{(3)}))\otimes m^{(2)}\big)\otimes\mu''\beta'S^{*-1}(\mu'),
\end{aligned}$$

which is the same. For B, similarly, the left-hand side of (1.4) is (assuming the precedence $ab\leftharpoonup\beta = (ab)\leftharpoonup\beta$, and so on)

$$\begin{aligned}
&((\mu''\otimes m')\triangleright b)_{(-1)}(\mu'\otimes m'')\otimes((\mu''\otimes m')\triangleright b)_{(0)}\\
&\quad= \big(\varepsilon\otimes\big((m'bS(m''))\leftharpoonup S^{*-1}(\mu'')\big)'\big)(\mu'\otimes m''')\otimes\big((m'bS(m''))\leftharpoonup S^{*-1}(\mu'')\big)''
\end{aligned}$$

$$= \big(\varepsilon \otimes \big((m'bS(m''))' \leftharpoonup S^{*-1}(\mu'')\big)\big)(\mu' \otimes m''') \otimes (m'bS(m''))''$$
(because $\Delta(a \leftharpoonup \mu) = (a' \leftharpoonup \mu) \otimes a''$)
$$= \big(\mu'' \otimes \big(S^{*-1}(\mu') \rightharpoonup (m'bS(m''))'\big)m'''\big) \otimes (m'bS(m''))''$$
(using the $\mathcal{D}(B)$-identity $\big(\varepsilon \otimes (b \leftharpoonup S^{*-1}(\mu''))\big)(\mu' \otimes 1) = \mu'' \otimes (S^{*-1}(\mu') \rightharpoonup b)$)
$$= \langle S^{*-1}(\mu'), m^{(2)}b''S(m^{(5)})\rangle \big(\mu'' \otimes (m^{(1)}b'S(m^{(6)}))m^{(7)}\big) \otimes m^{(3)}b'''S(m^{(4)})$$
$$= \big(\mu'' \otimes m^{(1)}b'\big) \otimes \big(m^{(2)}b''S(m^{(3)}) \leftharpoonup S^{*-1}(\mu')\big)$$
$$= \big((\mu'' \otimes m')(\varepsilon \otimes b')\big) \otimes \big((\mu' \otimes m'') \triangleright b''\big)$$
$$= \big((\mu \otimes m)'b_{(-1)}\big) \otimes \big((\mu \otimes m)'' \triangleright b_{(0)}\big),$$
which is the right-hand side.

2.2.5. Lemma. *$B^{*\mathrm{cop}}$ and B are braided commutative $\mathcal{D}(B)$-module algebras.*

This is entirely obvious once we note that when the $\mathcal{D}(B)$ action on $B^{*\mathrm{cop}}$ in **2.2.1** is restricted to the action of $B^{*\mathrm{cop}} \otimes 1$, it becomes the adjoint action; the same is true for the $\mathcal{D}(B)$ action on B restricted to the action of $\varepsilon \otimes B$; therefore, for example, $(a_{(-1)} \triangleright b)a_{(0)} = (a' \triangleright b)a'' = (a'bS(a''))a''' = ab$.

2.2.6. Lemma. *$B^{*\mathrm{cop}}$ and B are braided symmetric.*

We must show that $c_{B^{*\mathrm{cop}},B} = c^{-1}_{B,B^{*\mathrm{cop}}}$, i.e.,
$$(b_{(-1)} \triangleright \beta) \otimes b_{(0)} = \beta_{(0)} \otimes (S^{-1}_{\mathcal{D}}(\beta_{(-1)}) \triangleright b).$$
The antipode here is that of $\mathcal{D}(B)$, and therefore the right-hand side evaluates as $\beta' \otimes (S^*(\beta'') \triangleright b) = \beta' \otimes (b \leftharpoonup S^{*-1}(S^*(\beta''))) = \beta' \otimes (b \leftharpoonup \beta'')$, which is immediately seen to coincide with the left-hand side.

2.3. It now follows from **2.1.1** that $B^{*\mathrm{cop}} \bowtie B$ is a braided commutative Yetter–Drinfeld $\mathcal{D}(B)$-module algebra. But the product in $B^{*\mathrm{cop}} \bowtie B$ actually evaluates as the product in $\mathcal{H}(B^*)$:
$$(\alpha \bowtie a)(\beta \bowtie b) = \alpha(a_{(-1)} \triangleright \beta) \bowtie a_{(0)}b = \alpha((\varepsilon \otimes a') \triangleright \beta) \bowtie a''b = \alpha(a' \rightharpoonup \beta) \bowtie a''b.$$
We therefore conclude that *with the $\mathcal{D}(B)$ action and coaction in* (1.2) *and* (1.3), *$\mathcal{H}(B^*)$ is a braided commutative Yetter–Drinfeld $\mathcal{D}(B)$-module algebra*.

2.4. A “pseudo-adjoint” interpretation of the $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$. The action defined in (1.2) can be written in the “pseudo-adjoint” form
$$(\mu \otimes m) \triangleright (\alpha \# a) = (\mu'' \# m') \star (\alpha \# a) \star \mathsf{s}(\mu' \otimes m''), \tag{2.3}$$
where $\star$ temporarily denotes the composition in $\mathcal{H}(B^*)$, and
$$\mathsf{s}(\mu \otimes m) = (\varepsilon \# S(m)) \star (S^{*-1}(\mu) \# 1)$$

$$= (S(m'')\rightharpoonup S^{*-1}(\mu)) \# S(m').$$

The right-hand side of (2.3) is to be compared with the adjoint action of $\mathcal{D}(B)$ on itself,

$$(\mu\otimes m)\blacktriangleright(\nu\otimes n) = (\mu''\otimes m')\,(\nu\otimes n)\,S_{\mathcal{D}(B)}(\mu'\otimes m''),$$

where $S_{\mathcal{D}(B)}(\mu\otimes m) = (\varepsilon\otimes S(m))(S^{*-1}(\mu)\otimes 1) = (S(m''')\rightharpoonup S^{*-1}(\mu)\leftharpoonup m')\otimes S(m'')$.

2.4.1. To show (2.3), we calculate its right-hand side as

$$\begin{aligned}
&(\mu''\# m')\star(\alpha\# a)\star\big((S(m''')\rightharpoonup S^{*-1}(\mu'))\# S(m'')\big)\\
&\qquad= \big(\mu''(m^{(1)}\rightharpoonup\alpha)\# m^{(2)}a\big)\star\big((S(m^{(4)})\rightharpoonup S^{*-1}(\mu'))\# S(m^{(3)})\big)\\
&\qquad= \mu''(m^{(1)}\rightharpoonup\alpha)\big(m^{(2)}a'S(m^{(5)})\rightharpoonup S^{*-1}(\mu')\big)\# m^{(3)}a''S(m^{(4)})\\
&\qquad= \mu''(m'\rightharpoonup\alpha)\big((m''aS(m'''))'\rightharpoonup S^{*-1}(\mu')\big)\#(m''aS(m'''))''\\
&\qquad= \mu'''(m'\rightharpoonup\alpha)S^{*-1}(\mu'')\#(m''aS(m''')\leftharpoonup S^{*-1}(\mu'))
\end{aligned}$$

(because $(a'\rightharpoonup\mu)\otimes a'' = \mu'\otimes(a\leftharpoonup\mu'')$).

2.4.2. It may be interesting to see in more detail *why* the mock-adjoint action in (2.3) is a $\mathcal{D}(B)$ action. We recall from [6] that Eq. (1.7) holds for the product on $\mathcal{H}(B^*)$, with the $\mathcal{D}(B)$ product in the right-hand side and with the 2-cocycle $\eta:\mathcal{D}(B)\otimes\mathcal{D}(B)\to k$ given by

$$\eta(\mu\otimes m,\nu\otimes n) = \langle\mu,1\rangle\langle\nu,m\rangle\langle\varepsilon,n\rangle.$$

Of course, $(M,A)\mapsto M\star A$ is not a left action and $(M,A)\mapsto A\star\mathsf{s}(M)$ is not a right action of $\mathcal{D}(B)$; instead, we have the associativity of the $\star$ product, $M\star(A\star N) = (M\star A)\star N$ for all $M,A,N\in B^*\otimes B$. But the identity $\eta(M',N')\,\mathsf{s}(N'')\star\mathsf{s}(M'') = \mathsf{s}(MN)$ satisfied by Lu's cocycle η and the "pseudo-antipode" s ensures that (2.3) (i.e., (1.8)) is nevertheless a $\mathcal{D}(B)$ action.

From this perspective, furthermore, the $\mathcal{D}(B)$ module algebra property of $\mathcal{H}(B^*)$ is ensured by another "antipode-like" property of s, $\mathsf{s}(M')\star M'' = \varepsilon(M)1$, $M\in\mathcal{D}(B)$. And the Yetter–Drinfeld condition easily follows for the "pseudo-adjoint" action because $\delta\mathsf{s}(M) = S(M'')\otimes\mathsf{s}(M')$ (where δ is the same as $\Delta_{\mathcal{D}(B)}$ and the right-hand side is viewed as an element of $\mathcal{D}(B)\otimes\mathcal{H}(B^*)$) and, of course, because $\mathcal{H}(B^*)$ is a $\mathcal{D}(B)$ comodule algebra [6]. We somewhat formalize this simple argument as the following theorem (all of whose conditions hold for Lu's cocycle).

2.4.3. Theorem. *For a Hopf algebra (H,Δ,S,ε) with bijective antipode, let η be a normal right* 2*-cocycle* [6]*, i.e., a bilinear map $H\otimes H\to k$ such that*

$$\eta(f'g',h)\eta(f'',g'') = \eta(f,g'h')\eta(g'',h''),\qquad \eta(1,h) = \eta(h,1) = \varepsilon(h)$$

for all $f,g,h\in H$, and let $H_\star = (H,\star)$ denote the associative algebra with the product

$$g\star h = g'h'\eta(g'',h'').$$

Let $\mathsf{s} : H \to H$ *be given by*

$$\mathsf{s}(h) = \eta(h', h'')S(h'''). \tag{2.4}$$

If the conditions

$$\eta(\mathsf{s}(h'), h'') = \varepsilon(h), \tag{2.5}$$

$$\eta(h', \mathsf{s}(h'')) = \varepsilon(h), \tag{2.6}$$

$$\eta(g', h')\eta(\mathsf{s}(h''), \mathsf{s}(g'')) = \eta(g'h', g''h'') \tag{2.7}$$

hold for all $g, h \in H$*, then* $H_\star$ *is a left–left Yetter–Drinfeld* H*-module algebra under the left* H*-action*

$$g \triangleright h = g' \star h \star \mathsf{s}(g'') \tag{2.8}$$

and left coaction $\delta = \Delta$*, viewed as a map* $H_\star \to H \otimes H_\star$*. Moreover,* $H_\star$ *is braided commutative.*

Conditions (2.5)–(2.7) can be reformulated as

$$\mathsf{s}(h') \star h'' = \varepsilon(h)1, \tag{2.9}$$

$$h' \star \mathsf{s}(h'') = \varepsilon(h)1, \tag{2.10}$$

$$\eta(g', h')\mathsf{s}(h'') \star \mathsf{s}(g'') = \mathsf{s}(gh). \tag{2.11}$$

Also, it follows from (2.4) that $\Delta(\mathsf{s}(h)) = S(h'') \otimes \mathsf{s}(h')$.

That (2.8) is an H action immediately follows from (2.11). The module algebra property follows from (2.9). The left coaction δ makes $H_\star$ into a comodule algebra for any right cocycle η [6]. The Yetter–Drinfeld axiom is then verified as straightforwardly as for the true adjoint action:

$$\begin{aligned}(h' \triangleright g)_{(-1)}h'' \otimes (h' \triangleright g)_{(0)} &= (h' \star g \star \mathsf{s}(h''))'h''' \otimes (h' \star g \star \mathsf{s}(h''))'' \\ &= h^{(1)}g'S(h^{(4)})h^{(5)} \otimes h^{(2)} \star g'' \star \mathsf{s}(h^{(3)}) = h'g_{(-1)} \otimes (h'' \triangleright g_{(0)}).\end{aligned}$$

The braided commutativity is also immediate:

$$\begin{aligned}(h_{(-1)} \triangleright g) \star h_{(0)} &= h'_{(-1)} \star g \star \mathsf{s}(h''_{(-1)}) \star h_{(0)} = h' \star g \star \mathsf{s}(h'') \star h''' \\ &= h' \star g \star 1\varepsilon(h'') = h \star g.\end{aligned}$$

2.5. Multiple braided products. Further examples of Yetter–Drinfeld module algebras are produced by extending the Heisenberg double $\mathcal{H}(B^*)$ to multiple "alternating" braided products. We first return to the setting of **2.1**.

2.5.1. Multiple braided products $X_1 \bowtie \ldots \bowtie X_N$ of Yetter–Drinfeld H-module algebras X_i are the corresponding tensor products with the diagonal action and codiagonal coaction of H, and with the relations

$$x[i] \bowtie y[j] = (x_{(-1)} \triangleright y)[j] \bowtie x_{(0)}[i], \quad i > j, \tag{2.12}$$

where $z[i] \in X_i$. (The inverse relation is $x[i] \bowtie y[j] = y_{(0)}[j] \bowtie (S^{-1}(y_{(-1)}) \triangleright x)[i]$, $i < j$.) It readily follows from the Yetter–Drinfeld module algebra axioms for each of the X_i that $X_1 \bowtie \ldots \bowtie X_N$ is an associative algebra and, in fact, a Yetter–Drinfeld H-module algebra. In particular, it follows that

$$\begin{aligned}&(x_1[i_1] \bowtie \ldots \bowtie x_m[i_m]) \bowtie (y_1[j_1] \bowtie \ldots \bowtie y_n[j_n])\\ &\qquad = \big((x_{1_{(-1)}} \ldots x_{m_{(-1)}}) \triangleright (y_1[j_1] \bowtie \ldots \bowtie y_n[j_n])\big) \bowtie \big(x_{1_{(0)}}[i_1] \bowtie \ldots \bowtie x_{m_{(0)}}[i_m]\big)\end{aligned}$$

whenever $i_a > j_b$ for all $a = 1, \ldots, m$ and $b = 1, \ldots, n$.

2.5.2. "Alternating" braided products. Next, let X and Y be braided symmetric Yetter–Drinfeld H-module algebras, and consider the "alternating" products

$$X \bowtie Y \bowtie X \bowtie Y \bowtie \ldots,$$

with an arbitrary number of factors (or a similar product with the leftmost Y, or actually their inductive limits with respect to the obvious embeddings). We let $X[i]$ denote the ith copy of X, and similarly with $Y[j]$. For arbitrary $x[i] \in X[i]$ and $y[j] \in Y[j]$, we then have relations (2.12), i.e.,

$$x[2i+1] \bowtie y[2j] = (x_{(-1)} \triangleright y)[2j] \bowtie x_{(0)}[2i+1], \tag{2.13}$$

for all $i \geqslant j$, but by the braided symmetry condition, relations (2.13) — replicas of the relations between elements of X and elements of Y in $X \bowtie Y$ — *hold for all i and j*. In the multiple products, in addition, we also have the relations

$$\left.\begin{aligned} x[2i+1] \bowtie v[2j+1] &= (x_{(-1)} \triangleright v)[2j+1] \bowtie x_{(0)}[2i+1], \quad x, v \in X,\\ y[2i] \bowtie u[2j] &= (y_{(-1)} \triangleright u)[2j] \bowtie y_{(0)}[2i], \quad y, u \in Y, \end{aligned}\right. \quad i > j \tag{2.14}$$

(which also hold for $i = j$ if X and Y are braided commutative.)

2.5.3. Heisenberg n-tuples/chains. Generalizing $\mathcal{H}(B^*) \cong B^{*\mathrm{cop}} \bowtie B \cong B \bowtie B^{*\mathrm{cop}}$, we have "Heisenberg n-tuples/chains" — the alternating products

$$\mathfrak{H}_{2n} = B^{*\mathrm{cop}} \bowtie B \bowtie B^{*\mathrm{cop}} \bowtie B \bowtie \ldots \bowtie B,$$
$$\mathfrak{H}_{2n+1} = B^{*\mathrm{cop}} \bowtie B \bowtie B^{*\mathrm{cop}} \bowtie B \bowtie \ldots \bowtie B \bowtie B^{*\mathrm{cop}}.$$

As we saw in **2.5.2**, the following relations hold here:

$$b[2i]\,\beta[2j+1] = (b' \rightharpoonup \beta)[2j+1]\,b''[2i], \quad b \in B,\ \beta \in B^{*\mathrm{cop}},\ \text{ for all } i \text{ and } j$$

(where $B^{*\mathrm{cop}} \to B^{*\mathrm{cop}}[2j+1]$ and $B \to B[2i]$ are the morphisms onto the respective factors, and we omit the $\bowtie$ symbol for brevity), and

$$\alpha[2i+1]\,\beta[2j+1] = (\alpha'''\beta S^{*-1}(\alpha''))[2j+1]\,\alpha'[2i+1], \quad \alpha, \beta \in B^{*\mathrm{cop}}, \quad i \geqslant j,$$

$$a[2i]\,b[2j] = (a'bS(a''))[2j]\,a'''[2i], \qquad a,b\in B,\quad i\geqslant j.$$

The $\mathcal{D}(B)$ action is diagonal and the coaction is codiagonal, for example,

$$\begin{aligned}\delta(\alpha\bowtie a\bowtie\beta\bowtie b) &= \big((\alpha''\otimes 1)(\varepsilon\otimes a')(\beta''\otimes 1)(\varepsilon\otimes b')\big)\otimes\big(\alpha'\bowtie a''\bowtie\beta'\bowtie b''\big)\\ &= \big((\alpha''\otimes a')(\beta''\otimes b')\big)\otimes\big(\alpha'\bowtie a''\bowtie\beta'\bowtie b''\big).\end{aligned}$$

The chains with the leftmost B factor are defined entirely similarly. The obvious embeddings allow defining (one-sided or two-sided) inductive limits of alternating chains. All the chains are Yetter–Drinfeld module algebras, but none with $\geqslant 3$ factors are braided commutative in general.

3. Yetter–Drinfeld module algebras and the associated Hopf algebroid for $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$

In this section, we construct Yetter–Drinfeld module algebras for $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ at the $2p$th root of unity for an integer $p\geqslant 2$ (see **1.5**), and also consider the Hopf algebroid associated with a braided commutative Yetter–Drinfeld module algebra in accordance with the construction in [2].

$\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ can be obtained as a subquotient of the Drinfeld double of a Taft Hopf algebra B [23, 24] (a trick also used, e.g., in [43] for a closely related quantum group). On the "Heisenberg side," $\mathcal{H}(B^*)$ similarly yields $\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2)$, a $2p^3$-dimensional braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$-module algebra. This is worked out in **3.1–3.2** below; in **3.3**, dropping the coinvariants in $\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2)$, we obtain the algebra of $p\times p$ matrices, which is also a braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$-module algebra. In **3.4**, we use the Brzeziński–Militaru theorem to construct the corresponding Hopf algebroid. Multiple alternating braided products are considered in **3.5**.

3.1. $\mathcal{D}(B)$ and $\mathcal{H}(B^*)$ for the Taft Hopf algebra B.

3.1.1. The Taft Hopf algebra B. Let

$$B = \mathrm{Span}(E^m k^n),\quad 0\leqslant m\leqslant p-1,\quad 0\leqslant n\leqslant 4p-1,$$

be the $4p^2$-dimensional Hopf algebra generated by E and k with the relations

$$kE = \mathfrak{q}Ek,\quad E^p = 0,\quad k^{4p} = 1, \tag{3.1}$$

and with the comultiplication, counit, and antipode given by

$$\begin{gathered}\Delta(E) = 1\otimes E + E\otimes k^2,\quad \Delta(k) = k\otimes k,\quad \varepsilon(E) = 0,\quad \varepsilon(k) = 1,\\ S(E) = -Ek^{-2},\quad S(k) = k^{-1}.\end{gathered} \tag{3.2}$$

We define $F,\varkappa\in B^*$ by

$$\langle F, E^m k^n\rangle = \delta_{m,1}\frac{\mathfrak{q}^{-n}}{\mathfrak{q}-\mathfrak{q}^{-1}},\qquad \langle\varkappa, E^m k^n\rangle = \delta_{m,0}\mathfrak{q}^{-n/2}.$$

Then [23]

$$B^* = \mathrm{Span}(F^a \varkappa^b), \quad 0 \leqslant a \leqslant p-1, \quad 0 \leqslant b \leqslant 4p-1.$$

3.1.2. The Drinfeld double $\mathcal{D}(B)$. Direct calculation shows [23] that the Drinfeld double $\mathcal{D}(B)$ is the Hopf algebra generated by E, F, k, and $\varkappa$ with the relations given by

i) relations (3.1) in B,
ii) the relations $\varkappa F = \mathfrak{q} F \varkappa$, $F^p = 0$, and $\varkappa^{4p} = 1$ in B^*, and
iii) the cross-relations

$$k\varkappa = \varkappa k, \quad kFk^{-1} = \mathfrak{q}^{-1}F, \quad \varkappa E \varkappa^{-1} = \mathfrak{q}^{-1}E, \quad [E,F] = \frac{k^2 - \varkappa^2}{\mathfrak{q} - \mathfrak{q}^{-1}}.$$

The Hopf-algebra structure $(\Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}}, S_{\mathcal{D}})$ of $\mathcal{D}(B)$ is given by (3.2) and

$$\Delta_{\mathcal{D}}(F) = \varkappa^2 \otimes F + F \otimes 1, \quad \Delta_{\mathcal{D}}(\varkappa) = \varkappa \otimes \varkappa, \quad \varepsilon_{\mathcal{D}}(F) = 0, \quad \varepsilon_{\mathcal{D}}(\varkappa) = 1,$$
$$S_{\mathcal{D}}(F) = -\varkappa^{-2}F, \quad S_{\mathcal{D}}(\varkappa) = \varkappa^{-1}.$$

3.1.3. The Heisenberg double $\mathcal{H}(B^*)$. For the above B, $\mathcal{H}(B^*)$ is spanned by

$$F^a \varkappa^b \# E^c k^d, \qquad a, c = 0, \dots, p-1, \quad b, d \in \mathbb{Z}/(4p\mathbb{Z}), \tag{3.3}$$

where $\varkappa^{4p} = 1$, $k^{4p} = 1$, $F^p = 0$, and $E^p = 0$. A convenient basis in $\mathcal{H}(B^*)$ can be chosen as $(\varkappa, z, \lambda, \partial)$, where $\varkappa$ is understood as $\varkappa \# 1$ and

$$\begin{aligned} z &= -(\mathfrak{q} - \mathfrak{q}^{-1})\varepsilon \# E k^{-2}, \\ \lambda &= \varkappa \# k, \\ \partial &= (\mathfrak{q} - \mathfrak{q}^{-1}) F \# 1. \end{aligned}$$

The relations in $\mathcal{H}(B^*)$ then become $\varkappa z = \mathfrak{q}^{-1} z \varkappa$, $\varkappa \lambda = \mathfrak{q}^{\frac{1}{2}} \lambda \varkappa$, $\varkappa \partial = \mathfrak{q} \partial \varkappa$, $\varkappa^{4p} = 1$, and

$$\begin{aligned} &\lambda^{4p} = 1, \qquad z^p = 0, \qquad \partial^p = 0, \\ &\lambda z = z \lambda, \qquad \lambda \partial = \partial \lambda, \\ &\partial z = (\mathfrak{q} - \mathfrak{q}^{-1}) 1 + \mathfrak{q}^{-2} z \partial. \end{aligned} \tag{3.4}$$

Then the $\mathcal{D}(B)$ action on $\mathcal{H}(B^*)$ in (1.2) becomes $\varkappa \triangleright \varkappa^n = \varkappa^n$, $\varkappa \triangleright \partial^n = \mathfrak{q}^n \partial^n$, $\varkappa \triangleright \lambda^n = \mathfrak{q}^{\frac{n}{2}} \lambda^n$, $\varkappa \triangleright z^n = \mathfrak{q}^{-n} z^n$, and

$$\begin{aligned} &E \triangleright \varkappa = 0, && k \triangleright \varkappa^n = \mathfrak{q}^{-\frac{n}{2}} \varkappa, && F \triangleright \varkappa^n = -\mathfrak{q}^{\frac{n}{2}} [\tfrac{n}{2}] \partial \varkappa^n, \\ &E \triangleright \lambda^n = \mathfrak{q}^{-\frac{n}{2}} [\tfrac{n}{2}] \lambda^n z, && k \triangleright \lambda^n = \mathfrak{q}^{-\frac{n}{2}} \lambda, && F \triangleright \lambda^n = -\mathfrak{q}^{\frac{n}{2}} [\tfrac{n}{2}] \lambda^n \partial, \\ &E \triangleright z^n = -\mathfrak{q}^n [n] z^{n+1}, && k \triangleright z^n = \mathfrak{q}^n z^n, && F \triangleright z^n = [n] \mathfrak{q}^{1-n} z^{n-1}, \\ &E \triangleright \partial^n = \mathfrak{q}^{1-n} [n] \partial^{n-1}, && k \triangleright \partial^n = \mathfrak{q}^{-n} \partial^n, && F \triangleright \partial^n = -\mathfrak{q}^n [n] \partial^{n+1}. \end{aligned} \tag{3.5}$$

3.2. The $(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2), \overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2))$ pair.

3.2.1. From $\mathcal{D}(B)$ to $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$. The "truncation" whereby $\mathcal{D}(B)$ yields $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ consists

of two steps [23]: first, taking the quotient

$$\overline{\mathcal{D}(B)} = \mathcal{D}(B)/(\varkappa k - 1) \tag{3.6}$$

by the Hopf ideal generated by the central element $\varkappa \otimes k - \varepsilon \otimes 1$ and, second, identifying $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ as the subalgebra in $\overline{\mathcal{D}(B)}$ spanned by $F^{\ell} E^m k^{2n}$ with $\ell, m = 0, \dots, p-1$ and $n = 0, \dots, 2p-1$. It then follows that $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ is a Hopf algebra — the one described in **1.5**, where $K = k^2$.

The category of finite-dimensional $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ representations is not braided [28].

3.2.2. From $\mathcal{H}(B^*)$ to $\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2)$. In $\mathcal{H}(B^*)$, dually to the two steps just mentioned, we take a subalgebra and then a quotient [4]. In the basis chosen above, the subalgebra (which is also a $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ submodule) is the one generated by z, ∂, and λ. Its quotient by $\lambda^{2p} = 1$ gives the $2p^3$-dimensional algebra

$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = \mathbb{C}[z, \partial, \lambda] \big/ \big(\text{(3.4) and } (\lambda^{2p} - 1)\big).$$

As an associative algebra,

$$\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) = \mathbb{C}_{\mathfrak{q}}[z, \partial] \otimes \big(\mathbb{C}[\lambda]/(\lambda^{2p} - 1)\big),$$

with the p^2-dimensional algebra

$$\mathbb{C}_{\mathfrak{q}}[z, \partial] = \mathbb{C}[z, \partial]/(z^p,\ \partial^p,\ \partial z - (\mathfrak{q} - \mathfrak{q}^{-1}) - \mathfrak{q}^{-2} z \partial). \tag{3.7}$$

The $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ action on $\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2)$ is given by the last three lines in (3.5), with the central column rewritten for $K = k^2$. The coaction $\delta : \overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2) \to \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2) \otimes \overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2)$ follows from (1.3) as

$$\begin{aligned}
\lambda &\mapsto 1 \otimes \lambda, \\
z^m &\mapsto \sum_{s=0}^{m} (-1)^s \mathfrak{q}^{s(1-m)} (\mathfrak{q} - \mathfrak{q}^{-1})^s \begin{bmatrix} m \\ s \end{bmatrix} E^s K^{-m} \otimes z^{m-s}, \\
\partial^m &\mapsto \sum_{s=0}^{m} \mathfrak{q}^{s(m-s)} (\mathfrak{q} - \mathfrak{q}^{-1})^s \begin{bmatrix} m \\ s \end{bmatrix} F^s K^{s-m} \otimes \partial^{m-s}.
\end{aligned}$$

3.2.3. *With the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ action and coaction given above, $\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2)$ is a braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-module algebra.*

3.3. Matrix braided commutative Yetter–Drinfeld module algebras. It follows that $\mathbb{C}_{\mathfrak{q}}[z, \partial]$ in (3.7) — the algebra of "quantum differential operators on a line" — is also a braided commutative Yetter–Drinfeld $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-module algebra. It is in fact the full matrix algebra [39],

$$\mathbb{C}_{\mathfrak{q}}[z, \partial] \cong \mathrm{Mat}_p(\mathbb{C}). \tag{3.8}$$

3.3.1. That $\mathbb{C}_{\mathfrak{q}}[z,\partial]$ is (semisimple and) isomorphic to $\mathrm{Mat}_p(\mathbb{C})$ already follows from a more general picture elegantly developed in [44], where "para-Grassmann" algebras of the form $\mathbb{C}[z,\partial]/(z^p,\partial^p)$ with *various* additional relations on the $z^i\partial^j$ were studied. The relations between our z and ∂,

$$\partial^m z^n = \sum_{i\geqslant 0} \mathfrak{q}^{-(2m-i)n+im-\frac{i(i-1)}{2}} \begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} n\\ i\end{bmatrix}[i]!\,\big(\mathfrak{q}-\mathfrak{q}^{-1}\big)^i z^{n-i}\partial^{m-i}$$

(where the range of i is bounded above by $\min(m,n)$ because of the $\mathfrak{q}$-binomial coefficients), are nondegenerate in terms of the classification in [44], hence the isomorphism with the full matrix algebra.

We describe (3.8) as an isomorphism *of* $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ *module comodule algebras*. The generators z and ∂ have the respective matrix representations Z and D in (1.15) (where we do not reduce the expressions using that $\mathfrak{q}^p=-1$ and $[p-i]=[i]$ to highlight a pattern). Coaction (1.16) is then just the $m=1$ case of the formulas in **3.2.2**, and it is not difficult to see that the last three lines in (3.5) yield formulas (1.12)–(1.14) — so far, with no effect of the rescaling of λ in (1.12).

3.3.2. Once $\mathbb{C}_{\mathfrak{q}}[z,\partial]$ is thus identified with $\mathrm{Mat}_p(\mathbb{C})$, we can write

$$\overline{\mathcal{H}}_{\mathfrak{q}}s\ell(2) = \mathrm{Mat}_p\big(\mathbb{C}_{2p}[\lambda]\big)$$

(where we recall that $\mathbb{C}_{2p}[\lambda]=\mathbb{C}[\lambda]/(\lambda^{2p}-1)$), and it is immediate to see from (3.5) that λ entering the matrix entries rescales under the $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ action as indicated in (1.12). This establishes formulas (1.12)–(1.14).

3.3.3. For example, for $p=3$, choosing $x_{ij}=\lambda^{n_{ij}}y_{ij}$ with λ-independent y_{ij}, we have

$$F\triangleright\begin{pmatrix}\lambda^{n_{11}}y_{11} & \lambda^{n_{12}}y_{12} & \lambda^{n_{13}}y_{13}\\ \lambda^{n_{21}}y_{21} & \lambda^{n_{22}}y_{22} & \lambda^{n_{23}}y_{23}\\ \lambda^{n_{31}}y_{31} & \lambda^{n_{32}}y_{32} & \lambda^{n_{33}}y_{33}\end{pmatrix}$$
$$=\begin{pmatrix}\lambda^{n_{21}}y_{21} & \lambda^{n_{22}}y_{22}-\mathfrak{q}^{n_{11}}\lambda^{n_{11}}y_{11} & \lambda^{n_{23}}y_{23}+\mathfrak{q}^{n_{12}-2}\lambda^{n_{12}}y_{12}\\ \mathfrak{q}^{-1}\lambda^{n_{31}}y_{31} & \mathfrak{q}^{-1}\lambda^{n_{32}}y_{32}-\mathfrak{q}^{n_{21}-2}\lambda^{n_{21}}y_{21} & \mathfrak{q}^{-1}\lambda^{n_{33}}y_{33}+\mathfrak{q}^{n_{22}-4}\lambda^{n_{22}}y_{22}\\ 0 & -\mathfrak{q}^{n_{31}+2}\lambda^{n_{31}}y_{31} & \mathfrak{q}^{n_{32}}\lambda^{n_{32}}y_{32}\end{pmatrix}.$$

3.3.4. As an example of the coaction in matrix form, $\delta:\mathrm{Mat}_p(\mathbb{C})\to\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)\otimes\mathrm{Mat}_p(\mathbb{C})$, we give the only typographically manageable case, that of $p=2$. Writing elements of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)\otimes\mathrm{Mat}_2(\mathbb{C})$ as matrices with $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$-valued entries, we have

$$\delta X$$
$$=\begin{pmatrix}(1-2iEFK^3)x_{11}+Fx_{12}-2iEK^3x_{21}+2iEFK^3x_{22} & 2iEK^2x_{11}+K^3x_{12}-2iEK^2x_{22}\\ FK^3x_{11}+K^3x_{21}-FK^3x_{22} & (1-K^2-2iEFK^3)x_{11}+Fx_{12}-2iEK^3x_{21}+(K^2+2iEFK^3)x_{22}\end{pmatrix}.$$

3.3.5. It would be interesting to find a direct *matrix* derivation of the Yetter–Drinfeld axiom for $\mathrm{Mat}_p(\mathbb{C})$ and the braided commutativity property

$$(X_{(-1)} \triangleright Y)X_{(0)} = XY, \qquad X, Y \in \mathrm{Mat}_p(\mathbb{C}).$$

We illustrate the structure occurring in the left-hand side here before the matrix multiplication, with the known result, is evaluated (again, necessarily restricting ourself to $p = 2$):

$$\begin{aligned}(X_{(-1)} \triangleright Y) \otimes X_{(0)} = {}& \begin{pmatrix} y_{11} & -y_{12} \\ -y_{21} & y_{22} \end{pmatrix} \otimes \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \otimes \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} \\ & + \begin{pmatrix} -\frac{i}{2}y_{12} & 0 \\ \frac{i}{2}(y_{11}-y_{22}) & -\frac{i}{2}y_{12} \end{pmatrix} \otimes \begin{pmatrix} 0 & 2i(x_{11}-x_{22}) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{i}{2}y_{12} & 0 \\ \frac{i}{2}(y_{11}-y_{22}) & \frac{i}{2}y_{12} \end{pmatrix} \otimes \begin{pmatrix} -2ix_{21} & 0 \\ 0 & -2ix_{21} \end{pmatrix} \\ & + \begin{pmatrix} y_{21} & y_{11}-y_{22} \\ 0 & y_{21} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x_{22}-x_{11} & 0 \end{pmatrix} + \begin{pmatrix} y_{21} & y_{22}-y_{11} \\ 0 & y_{21} \end{pmatrix} \otimes \begin{pmatrix} x_{12} & 0 \\ 0 & x_{12} \end{pmatrix} \\ & + \begin{pmatrix} \frac{i}{2}(y_{22}-y_{11}) & 0 \\ 0 & \frac{i}{2}(y_{22}-y_{11}) \end{pmatrix} \otimes \begin{pmatrix} 2i(x_{11}-x_{22}) & 0 \\ 0 & 2i(x_{11}-x_{22}) \end{pmatrix}.\end{aligned}$$

3.4. Hopf algebroid with the $\mathrm{Mat}_p(\mathbb{C})$ **base.** Theorem 4.1 in [2] nicely reinterprets the structure of a braided commutative Yetter–Drinfeld H-module algebra A as a bialgebroid structure on $A \# H$. (We refer the reader to [2] for a comprehensive discussion of (Hopf|bi)algebroids, also in relation to Lu's bialgebroids [16], Xu's bialgebroids with an anchor [45], and Takeuchi's $\times_A$-bialgebras [46], as well as for references to other related works.)

The examples of Hopf algebroids $A \# H$ with our braided commutative Yetter–Drinfeld module algebras $A = \mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda])$ or $A = \mathrm{Mat}_p(\mathbb{C})$ may be of some interest because of the explicit matrix structure of the base algebra A. Below, we follow [2], adapting the formulas there to a *left* comodule algebra by duly inserting the antipodes. To somewhat simplify the notation, we discuss the "λ-independent" example, i.e., the Hopf algebroid structure of $\mathcal{A} = \mathrm{Mat}_p(\mathbb{C}) \# \overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$; reintroducing $\mathbb{C}_{2p}[\lambda]$ on the matrix side is left to the reader.

As a vector space, $\mathcal{A} \cong \mathrm{Mat}_p(\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2))$, matrices with $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$-valued entries; we can therefore write $1 \# h = \mathbf{1}h$ $(h \in \overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2))$, where $\mathbf{1}$ is the unit $p \times p$ matrix; with a slight abuse of notation, similarly, $X \# 1 = X$, understood as a "constant" $p \times p$ matrix. An arbitrary element of $\mathcal{A}$ can be written as $\sum_{i,j=1}^{p} \mathsf{e}_{ij} h_{ij}$, where the e_{ij} are the standard elementary matrices and $h_{ij} \in \overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$. The smash-product composition is then given by

$$\Big(\sum_{i,j=1}^{p} \mathsf{e}_{ij} h_{ij}\Big)\Big(\sum_{m,n=1}^{p} \mathsf{e}_{mn} g_{mn}\Big) = \sum_{i,j=1}^{p} \sum_{m,n=1}^{p} \mathsf{e}_{ij} (h'_{ij} \triangleright \mathsf{e}_{mn})\, h''_{ij} g_{mn}, \quad h_{ij}, g_{mn} \in \overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2),$$

with the left action $\triangleright$ to be evaluated in accordance with (1.12)–(1.14). We write $\mathcal{A} = \mathrm{Mat}_p(\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2))_{\#}$, with the subscript reminding of the smash-product composition in this algebra (which is highly nonstandard from the matrix standpoint).

The relevant structures

$$\begin{aligned}\varepsilon &: \mathcal{A} \to \mathrm{Mat}_p(\mathbb{C}),\\ s,t &: \mathrm{Mat}_p(\mathbb{C}) \to \mathcal{A},\\ \Delta &: \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{Mat}_p(\mathbb{C})} \mathcal{A},\\ \tau &: \mathcal{A} \to \mathcal{A}\end{aligned}$$

(the counit, the source and target maps, the coproduct, and the antipode) are as follows.

The counit $\varepsilon : \mathcal{A} \to \mathrm{Mat}_p(\mathbb{C})$ acts componentwise,

$$\varepsilon\Big(\sum_{i,j=1}^{p} \mathsf{e}_{ij} h_{ij}\Big) = \sum_{i,j=1}^{p} \mathsf{e}_{ij}\varepsilon(h_{ij}).$$

The source map $s : \mathrm{Mat}_p(\mathbb{C}) \to \mathcal{A}$ is the identical map onto constant matrices. The target map $t : \mathrm{Mat}_p(\mathbb{C}) \to \mathcal{A} = \mathrm{Mat}_p(\mathbb{C}) \# \overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ is given by

$$t(X) = X_{(0)} \# S^{-1}(X_{(-1)}),$$

where $\delta(X) = X_{(-1)} \otimes X_{(0)} \in \overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2) \otimes \mathrm{Mat}_p(\mathbb{C})$ is the coaction defined in (1.16). It then follows that Z and D in (1.15) map under t into the following two-diagonal matrices with $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$-valued entries:

$$t(Z) = \begin{pmatrix} (\mathfrak{q}-\mathfrak{q}^{-1})E & 0 & & & \\ K & (\mathfrak{q}-\mathfrak{q}^{-1})E & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & K & (\mathfrak{q}-\mathfrak{q}^{-1})E & 0 \\ 0 & \dots\dots\dots\dots & & K & (\mathfrak{q}-\mathfrak{q}^{-1})E \end{pmatrix}, \tag{3.9}$$

$$t(D) = (\mathfrak{q}-\mathfrak{q}^{-1}) \begin{pmatrix} -FK & K & & & \\ 0 & -FK & \mathfrak{q}^{-1}[2]K & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -FK & \mathfrak{q}^{2-p}[p-1]K \\ 0 & \dots\dots & & 0 & -FK \end{pmatrix}. \tag{3.10}$$

For any complex matrix $Y = \sum_{m,n} y_{mn} Z^m D^n$, we use (3.9) and (3.10) to calculate $t(Y) = \sum_{m,n} y_{mn} t(D)^n t(Z)^m \in \mathcal{A}$ (evidently, with the smash-product multiplication understood). Furthermore, elementary calculation using the braided commutativity shows that

$$t(Y)(X \# h) = X \cdot t(Y) \cdot h, \quad X,Y \in \mathrm{Mat}_p(\mathbb{C}),$$

where, abusing the notation, the dot denotes both *matrix product and the product in* $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$, with articulately no "smash" effects because multiplication with a constant matrix is on the left and with a $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ element on the right.

The coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{Mat}_p(\mathbb{C})} \mathcal{A}$ is (co)componentwise,

$$\Delta\Big(\sum_{i,j=1}^{p} \mathsf{e}_{ij} h_{ij}\Big) = \sum_{i,j=1}^{p} \mathsf{e}_{ij} h'_{ij} \otimes_{\mathrm{Mat}_p(\mathbb{C})} \mathbf{1} h''_{ij}, \quad h_{ij} \in \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2).$$

The $\otimes_{\mathrm{Mat}_p(\mathbb{C})}$ product is here defined with respect to the right action of $\mathrm{Mat}_p(\mathbb{C})$ on $\mathcal{A}$ via $(X \# h).Y = t(Y)(X \# h)$ and the left action via $Y.(X \# h) = s(Y)(X \# h)$, and hence

$$(X \cdot t(A) \cdot h) \otimes_{\mathrm{Mat}_p(\mathbb{C})} (Y \# g) = (X \# h) \otimes_{\mathrm{Mat}_p(\mathbb{C})} (AY \# g)$$

holds for all $X, A, Y \in \mathrm{Mat}_p(\mathbb{C})$ and $g, h \in \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ (where the first factor in the left-hand side, again, involves matrix and $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ products on the different sides of $t(A)$).

The antipode $\tau : \mathcal{A} \to \mathcal{A}$ is given by another simple adaptation of a formula in [2]:

$$\tau(X \# h) = (1 \# S(h))\big((S(X''_{(-1)}) \triangleright X_{(0)}) \# S(X'_{(-1)})\big), \quad X \in \mathrm{Mat}_p(\mathbb{C}), \;\; h \in \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2),$$

with the product in the right-hand side to be taken in $\mathcal{A}$.[2] On $1 \# \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, this is just the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ antipode, and on $\mathrm{Mat}_p(\mathbb{C})$, $\tau(X) = (1 \# S(X_{(-1)}))(X_{(0)} \# 1)$; a simple calculation then shows that

$$\tau(Z) = \mathfrak{q}^2 t(Z),$$
$$\tau(D) = \mathfrak{q}^{-2} t(D).$$

Being an anti-algebra map, again, this extends to all of $\mathrm{Mat}_p(\mathbb{C}) \ni \sum_{m,n} y_{mn} Z^m D^n$.

Some of the Hopf algebroid properties (see [2, Defnition 2.2] for a nicely refined list of axioms), e.g., $\tau(t(X)) = s(X)$ and $t(X)s(Y) = s(Y)t(X)$, are evident for $\mathcal{A} = \mathrm{Mat}_p(\mathbb{C}) \# \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ described in matrix form; with others, it is not entirely obvious how far one can proceed with verifying them in a purely *matrix* language, i.e., not following [2] in resorting to the Yetter–Drinfeld module algebra properties; so much more interesting is the fact that $\mathrm{Mat}_p(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2))_{\#}$ at $\mathfrak{q} = e^{i\pi/p}$ *is* a Hopf algebroid over $\mathrm{Mat}_p(\mathbb{C})$.

As already noted, it is entirely straightforward to extend the above formulas to describe $\mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda]) \# \overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2) = \mathrm{Mat}_p(\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2) \otimes \mathbb{C}_{2p}[\lambda])_{\#}$ as a Hopf algebroid over $\mathrm{Mat}_p(\mathbb{C}_{2p}[\lambda]) \equiv \mathrm{Mat}_p(\mathbb{C}[\lambda]/(\lambda^{2p} - 1))$.

3.5. Heisenberg “chains.” The Heisenberg n-tuples/chains defined in **2.5.3** can also be “truncated” similarly to how we passed from $\mathcal{H}(B^*)$ to $\overline{\mathcal{H}}_{\mathfrak{q}} s\ell(2)$. An additional possibility

[2] And the section γ of the natural projection $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{Mat}_p(\mathbb{C})} \mathcal{A}$, required in the definition of a Hopf algebroid [2] to satisfy the condition $m \circ (\mathrm{id} \otimes \tau) \circ \gamma \circ \Delta = s \circ \varepsilon$, is given by $\gamma : (X \# h) \otimes_{\mathrm{Mat}_p(\mathbb{C})} (Y \# g) \mapsto (X \cdot t(Y) \cdot h) \otimes (\mathbf{1} g)$.

here is to drop the coinvariant λ altogether, which leaves us with the "*truly Heisenberg*" Yetter–Drinfeld $\overline{\mathscr{U}}_{\mathfrak{q}}s\ell(2)$-module algebras

$$\begin{aligned}\mathsf{H}_2 &= \mathbb{C}^{*p}_{\mathfrak{q}}[\partial_1] \bowtie \mathbb{C}^{p}_{\mathfrak{q}}[z_2] = \mathbb{C}_{\mathfrak{q}}[z_2,\partial_1],\\ \mathsf{H}_{2n} &= \mathbb{C}^{*p}_{\mathfrak{q}}[\partial_1] \bowtie \mathbb{C}^{p}_{\mathfrak{q}}[z_2] \bowtie \ldots \bowtie \mathbb{C}^{*p}_{\mathfrak{q}}[\partial_{2n-1}] \bowtie \mathbb{C}^{p}_{\mathfrak{q}}[z_{2n}],\\ \mathsf{H}_{2n+1} &= \mathbb{C}^{*p}_{\mathfrak{q}}[\partial_1] \bowtie \mathbb{C}^{p}_{\mathfrak{q}}[z_2] \bowtie \ldots \bowtie \mathbb{C}^{*p}_{\mathfrak{q}}[\partial_{2n-1}] \bowtie \mathbb{C}^{p}_{\mathfrak{q}}[z_{2n}] \bowtie \mathbb{C}^{*p}_{\mathfrak{q}}[\partial_{2n+1}]\end{aligned}$$

(or their infinite versions), where $\mathbb{C}^{*p}_{\mathfrak{q}}[\partial] = \mathbb{C}[\partial]/\partial^p$ and $\mathbb{C}^{p}_{\mathfrak{q}}[z] = \mathbb{C}[z]/z^p$ with the braiding inherited from **2.5.3**, which amounts to using the relations

$$\partial_i z_j = \mathfrak{q} - \mathfrak{q}^{-1} + \mathfrak{q}^{-2} z_j \partial_i$$

for *all* (odd) i and (even) j, and

$$\left.\begin{aligned} z_i z_j &= \mathfrak{q}^{-2} z_j z_i + (1-\mathfrak{q}^{-2}) z_j^2,\\ \partial_i \partial_j &= \mathfrak{q}^{2} \partial_j \partial_i + (1-\mathfrak{q}^{2}) \partial_j^2, \end{aligned}\right. \qquad i \geqslant j$$

(and $z_i^p =$ and $\partial_i^p = 0$; our relations may be interestingly compared with those in para-Grassmann algebras studied in [47]).

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